The Weisfeiler-Leman dimension of graphs, old and new results

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General remarks

The WL-dimension of a graph: origins

The Weisfeiler-Leman dimension $\dim_{\mathrm{WL}}(X)$ of an n-vertex graph X shows how easy it is to identify X up to isomorphism:

easy if $\dim_{\mathrm{WL}}(X) = 1$ and difficult if $\dim_{\mathrm{WL}}(X) \cong n$.

The Graph Isomorphism Problem (GIP)

The concept goes back to the WL-algorithm (WL, 1968) to test graph isomorphism; a general version of this algorithm used by L. Babai (2016) for solving GIP in quasipolynomial time.

The WL-dimension of a graph: at present

Formally, the WL-dimension of a graph was defined by M. Grohe (2017) in the framework of descriptive complexity, and later was generalized to groups.

The WL-dimension: mathematical logic, 1

The first order language for graphs

- variables x, y, $\ldots$,
- relations D (adjacency) and $=$,
- connectives $\wedge$, $\vee$, $\neg$, $\rightarrow$,
- quantifiers $\forall$ and $\exists$.

Example

A graph X satisfying the formula φ ($X \models \varphi$),

$$\varphi \equiv \forall x \, \forall y \, [D(x, y) \rightarrow (D(y, x) \wedge x \neq y)],$$

is undirected and has no loops.

The WL-dimension: mathematical logic, 2

Counting quantifiers

For each $i \in \mathbb{N}$ the formula $(\exists\, ix)\varphi(x)$ means that a graph has $\geq i$ vertices for which the formula φ is true.

Example

The formula $(\exists 17\, x)(\exists 5\, y)D(x, y)$ means that the graph has $\geq$ 17 vertices of degree $\geq$ 5.

The language $\mathfrak{C}_m$

For each $m \in \mathbb{N}$, the language $\mathfrak{C}_m$ consists of all formulas with counting quantifiers that have at most m distinct variables.

The WL-dimension: mathematical logic, 3

Definition (Grohe, 2017)

Two graphs X and X' are $\mathfrak{C}_m$-equivalent, $X \sim_{\mathfrak{C}_m} X'$, if for all formulas $\varphi \in \mathfrak{C}_m$, we have

$$X \models \varphi \quad \Leftrightarrow \quad X' \models \varphi.$$

The WL-dimension $\dim_{\mathrm{WL}}(X)$ of a graph X is defined to be the minimal number $m \in \mathbb{N}$ such that

$$X \sim_{\mathfrak{C}_m} X' \quad \Rightarrow \quad X \cong X'.$$

Remark

$1 \leq \dim_{\mathrm{WL}}(X) \leq n$ for every n-vertex graph X.

The WL-dimension: graph theory 1

Graphs of bounded tree-width

A graph X has tree-width $\leq m$ if there is a tree T such that

- any vertex of T is a vertex subset of X of size $\leq m + 1$,
- any edge of X is a subset of a vertex of T,
- the vertices of T containing a common vertex of X induce a connected subgraph of T.

Let $\mathfrak{S}_m$ be the set of all graphs of tree-width at most m.

Notation

- For graphs X, Y, let $\mathrm{hom}(Y, X)$ be the number of homomorphisms from Y to X.
- For a set $\mathfrak{S}$ of graphs, let $\mathrm{hom}_{\mathfrak{S}}(X)$ be the vector with entries $\mathrm{hom}(Y, X)$, $Y \in \mathfrak{S}$.

The WL-dimension: graph theory 2

Theorem [Z. Dvořák, 2010]

$X \sim_{\mathfrak{C}_m} X' \quad \Leftrightarrow \quad \mathrm{hom}_{\mathfrak{S}_m}(X) = \mathrm{hom}_{\mathfrak{S}_m}(X')$.

Remark

When the graphs in question have fixed order, the set $\mathfrak{S}_m$ can be replaced by its finite subset.

Corollary

$\dim_{\mathrm{WL}}(X) = \min\{m \in \mathbb{N}$: the vector $\mathrm{hom}_{\mathfrak{S}_m}(X)$ determines the graph X up to isomorphism$\}$.

The WL-dimension and the WL-algorithm, 1

For $v = (v_1, \ldots, v_m) \in V^m$, $w \in V$, and $1 \le i \le m$, set

$$v_{i \leftarrow w} = (v_1, \ldots, v_{i-1}, w, v_{i+1}, \ldots, v_m).$$

The m-dim WL-algorithm ($m \ge 2$)

Input: a graph X with vertex set V,
Output: an invariant coloring c of the m-tuples in V^m.

1. Set the initial coloring: $c(v)$, $v \in V^m$, is equal to "the isomorphism type" of the induced subgraph $X_{\{v_1,\ldots,v_m\}}$.
2. For each $v \in V^m$, find a multiset

$$s(v) = \{\!\{(c(v_{1\leftarrow w}), \ldots, c(v_{m\leftarrow w})) : \ w \in V\}\!\}.$$

3. Define a coloring c': $c'(u) < c'(v)$ iff $c(u) \prec c(v)$, or $c(u) = c(v)$ and $s(u) \prec s(v)$.
4. Go to step 2 if $|c| \ne |c'|$; otherwise, output c.

The WL-dimension and the WL-algorithm, 2

Definition

The m-dim WL-algorithm distinguishes graphs X and X' if there is a color i with $|c_X^{-1}(i)| \neq |c_{X'}^{-1}(i)|$, where c_X and $c_{X'}$ are the colorings obtained by the m-dim WL applied to X and X'.

Theorem (Cai-Fürer-Immerman, 1992)

- X and X' are not distinguished by the m-dim WL-algorithm if and only if X and X' are $\mathfrak{C}_{m+1}$-equivalent,
- there are pairs of non-isomorphic but $\mathfrak{C}_{m+1}$-equivalent graphs with $m = O(n)$.

Corollary

$\dim_{\mathrm{WL}}(X) = \min\{m \in \mathbb{N}$: the m-dim WL-algorithm distinguishes X from any nonisomorphic graph$\}$.

General facts and applications

Proposition

The isomorphism of n-vertex graphs of bounded WL-dimension can be tested in time polynomial in n.

Remarks

- the WL-dimension of a random graph is equal to 1,
- there is $0 < c < 1$ such that for sufficiently large n, there are n-vertex graphs X with $\dim_{\mathrm{WL}}(X) \geq cn$ (CFI, 1992),
- for every graph X, $\dim_{\mathrm{WL}}(X) \leq 0.15 \cdot n + o(n)$ (SS, 2024).

Proposition

Let X be a strongly regular graph. Then $\dim_{\mathrm{WL}}(X) \leq 2$ iff X is uniquely determined (up to isomorphism) by its parameters.

Bounded Weisfeiler-Leman dimension: special cases

$\dim_{\mathrm{WL}}(X) = 1$

The graphs X with $\dim_{\mathrm{WL}}(X) = 1$ have been completely characterized by Kiefer et al. (2015) and independently by Arvind et al. (2017). In particular, if X is a regular graph, then

$$\dim_{\mathrm{WL}}(X) = 1 \quad \Leftrightarrow \quad X \in \{K_n, \overline{K_n},\ nK_2,\ \overline{nK_2},\ C_5\}.$$

$\dim_{\mathrm{WL}}(X) \geq 2$

The characterization here is hopeless, because includes a characterization of strongly regular graphs uniquely determined by parameters. A special case concerning colored graphs with color classes of size ≤ 4 was studied by Fuhlbrück et al. (2021).

Small Weisfeiler-Leman dimension

Definition

Let $\mathfrak{K}$ be a class of graphs. Put

$$\dim_{\mathrm{WL}}(\mathfrak{K}) = \max_{X \in \mathfrak{K}} \dim_{\mathrm{WL}}(X).$$

Some classes with small Weisfeiler-Leman dimension

- trees: $\dim_{\mathrm{WL}}(\mathfrak{K}) = 1$,
- interval graphs: $\dim_{\mathrm{WL}}(\mathfrak{K}) = 2$ (Evdokimov et al., 2000),
- distance-hereditary graphs: $\dim_{\mathrm{WL}}(\mathfrak{K}) = 2$ (GNP, 2023),
- planar graphs: $\dim_{\mathrm{WL}}(\mathfrak{K}) \leq 3$ (Kiefer et al., 2017).

Large Weisfeiler-Leman dimension

Notation

Let $\mathfrak{K}$ be a class of graphs, $p : \mathfrak{K} \to \mathbb{N}$ a graph invariant, and r a positive integer. Put $\mathfrak{K}_{p,r} = \{X \in \mathfrak{K} : \ p(X) \le r\}$.

Some classes with bounded Weisfeiler-Leman dimension

- if $p(X)$ is the Hadwiger number of a graph X, then $\dim_{\mathrm{WL}}(\mathfrak{K}_{p,r}) \le f(r)$ for some function $f(\cdot)$ (Grohe, 2017),
- if $p(X)$ is the rank width of a graph X, then $\dim_{\mathrm{WL}}(\mathfrak{K}_{p,r}) \le 3r + 4$ (GN, 2023).

Remark

If $p(X)$ is the maximal degree of a graph X, then $\dim_{\mathrm{WL}}(\mathfrak{K}_{p,3}) = \infty$ (Cai et.al, 1992).

Further directions

Lower bounds for the *WL*-dimension

In the class $\mathfrak{K}_n$ of all *n*-vertex graphs, $\dim_{\mathrm{WL}}(\mathfrak{K}_n) \geq 0.00465 \cdot n$ (PV, 2011). It would be interesting to find proper subclasses of $\mathfrak{K}_n$ with non-constant lower bound for the WL-dimension.

Recognizing graph properties by the *m*-dim WL

A graph property $\mathcal{P}$ is WL_m*-invariant* if any two WL_m-equivalent graphs satisfy or not $\mathcal{P}$ simultaneously (Arvind et al., 2020). Isospectrality is WL_2-invariant. How about other properties?

The WL-dimension of groups

The WL-dimension for groups is defined as for graphs (BS, 2020). No class $\mathfrak{G}$ of groups is known for which $\dim_{\mathrm{WL}}(\mathfrak{G}) = \infty$.

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